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2-level fractional factorial designs which are the union of non trivial regular designs ⋆

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Abstract

Every fraction of a 2-level factorial design is a union of points, each of them being trivially a regular fraction. In order to find non-trivial decomposition, we derive a condition for the inclusion of a regular fraction in a generic fraction as follows. Regular fractions are characterized by a polynomial indicator function of the form $R = \frac{1}{l}\sum_{\alpha\in\mathcal{L}} e_\alpha X^\alpha$, where $\alpha \mapsto e_\alpha$ is a group homeomorphism from $\mathcal{L} \subset \mathbb{Z}_2^d$ into $\{-1,+1\}$ (Fontana et al., 2000). A regular fraction R is a subset of a given fraction $F = \sum_\alpha b_\alpha X^\alpha$ if $FR = R$, which in turn is equivalent to $\sum_t F(t)R(t) = \sum_t R(t)$. If $\mathcal{H} = \{\alpha_1 \dots \alpha_k\}$ is a generating set of $\mathcal{L}$, and $R = \frac{1}{2^k}(1 + e_1 X^{\alpha_1})\cdots(1 + e_k X^{\alpha_k})$, $e_j = \pm 1$, $j = 1 \dots k$, the inclusion condition in term of the b_α's is $b_0 + e_1 b_{\alpha_1} + \cdots + e_1 \cdots e_k b_{\alpha_1+\cdots+\alpha_k} = 1$. The practical applicability of the previous condition is discussed in the second part of the paper.

Key words: 2-level factorial design, Plackett-Burman design, Algebraic statistics, Indicator polynomial.

1 Introduction

We consider 2-level fractional designs with m factors, where the levels of each factor are coded $-1, +1$. The full factorial design is $\mathcal{D} = \{-1,+1\}^m$ and a fraction of the full design is a subset $\mathcal{F} \subset \mathcal{D}$. According to the algebraic description of designs, as it is discussed in Pistone et al. (2001) and Pistone et al. (2007), the *fraction ideal* Ideal$(\mathcal{F})$, also called design ideal, is the set of all polynomials with real coefficients that are zero on all points of the fraction.

⋆ This paper is an offspring of the Alcotra 158 EU research contract on the planning of sequential designs for sample surveys in tourism statistics. A preliminary version has been presented by R. Fontana at the DAE 2007 Conference, The University of Memphis, November 2, 2007

Two polynomials f and g are *aliased* by $\mathcal{F}$ if and only if $f-g \in \operatorname{Ideal}(\mathcal{F})$. The quotient linear space defined in such a way is the vector space of real responses on $\mathcal{F}$. The fraction ideal is generated by a finite number of its elements. This finite set of polynomials is called a *basis* of the ideal. Ideal bases are not uniquely determined, unless very special conditions are met. A *Gröbner basis* of the fraction ideal can be defined after the assignment of a total order on monomials called *monomial order*. If a monomial order is given, it is possible to identify the *leading monomial* of each polynomial. As far as applications to statistics are concerned, a Gröbner basis is characterized by the following property: the set of all monomials that are are not divided by any of the leading term of the polynomials in the basis form a linear basis of the quotient vector space. A general reference to the relevant computational commutative algebra topics is Cox et al. (1997) or Kreuzer and Robbiano (2000).

The ring of polynomials in m indeterminate $x_1 \dots x_m$ and rational coefficient is denoted by $R = \mathbb{Q}[x_1 \dots x_m]$. The design ideal $\operatorname{Ideal}(\mathcal{D})$ has a unique 'minimal' basis $x_1^2-1, \dots, x_m^2-1$, which happens to be a Gröbner basis. The polynomials that are added to this basis to generate the ideal of a fraction are called *generating equations.* An ideal with a basis of binomials with coefficients ± 1 is called *binomial ideal.* Indicator polynomials polynomials of a fraction were introduced in Fontana et al. (2000), see also Pistone and Rogantin (2008). An indicator polynomial has the form

$$F = \sum_{\alpha} b_\alpha x^\alpha, \quad \alpha = (\alpha_1, \dots, \alpha_m) \in \{0,1\}, \quad x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d} \tag{1}$$

and it satisfies the conditions $F(a) = 1$ if $a \in \mathcal{F}$, $F(a) = 0$ otherwise. If necessary, we distinguish between the indeterminate x_j, the value a_j and the mapping $X_j(a) = a_j$. How to move between the ideal representation and the indicator function representation, is discussed in Notari et al. (2007).

The definition and characterization, from the algebraic point of view, of regular fractional factorial designs (briefly regular designs) is discussed in Fontana et al. (2000), see also Pistone and Rogantin (2008). In particular, the last paper referred to considers mixed factorial design, but this case is outside the scope of the present paper. Orthogonal arrays as are defined in Hedayat et al. (1999) can be characterized in the previous algebraic framework, see Pistone and Rogantin (2008), as follows. A fraction $\mathcal{F}$ with indicator polynomial F is orthogonal with strength s if $b_\alpha = 0$ if $1 \le |\alpha| \le s$, $|\alpha| = \sum_j \alpha_j$. The notion of indicator polynomial can be accommodated to cases with replicated design points by allowing integer values other than 0 and 1 to F, see Ye (2003). In such a case, we prefer to call F a *counting polynomial* of the fraction. A systematic algebraic search of orthogonal arrays with replications is discussed in Carlini and Pistone (2007). For sake of easy reference in Section 5 below, we quote a couple of specific result about orthogonal arrays. In fact, considering $m = 5$ factors and strength $s = 2$, it is shown in (Carlini and Pistone, 2007,

Table 5.2) that there are 192 OA's with 12 points and no replications, and there are 32 OA's with 12 points, one of them replicated, then the support has 11 points.

This paper is organized as follows. In Section 2 the polynomial representation of regular fractions is reviewed. In Sections 3 and 4, such a polynomial representation is applied to the problem of finding fractions that are union of regular fractions. In Section 5 the important case of Plackett-Burman designs is considered.

2 Regular fractions

According to the definition used in Fontana et al. (2000) and Pistone and Rogantin (2007), a regular fraction is defined as follows. Let $\mathcal{L}$ be a subset of $L = \mathbb{Z}_2^m$, which is an additive group. Let Ω_2 be the multiplicative group $\{+1, -1\}$

Definition 2.1 *Let e be a map from $\mathcal{L}$ to Ω_2. A non-empty fraction $\mathcal{F}$ is* regular *if*

(1) $\mathcal{L}$ is a sub-group of $\subset L$; and
(2) the system of equations

$$X^\alpha = e(\alpha) \quad , \qquad \alpha \in \mathcal{L}$$

defines the fraction $\mathcal{F}$, i.e. such equations are a set of generating equations.

It follows from the previous conditions that the mapping e is in fact a group homeomorphism.

Other definitions are known to be equivalent to this one. We collected them in the following Theorem.

Theorem 2.1 *Let $\mathcal{F}$ be a fraction. The following statements are equivalent:*

(1) The fraction $\mathcal{F}$ is regular according to definition 2.1.
(2) The indicator function of the fraction has the form

$$F(\zeta) = \frac{1}{l} \sum_{\alpha \in \mathcal{L}} e(\alpha) \ X^\alpha(\zeta), \qquad \zeta \in \mathcal{D}.$$

where $\mathcal{L}$ is a given subset of L and $e : \mathcal{L} \rightarrow \Omega_2$ is a given mapping.
(3) For each $\alpha, \beta \in L$ the interactions represented on $\mathcal{F}$ by the terms X^α and X^β are either orthogonal or totally aliased.

(4) The Ideal $(\mathcal{F})$ *is binomial.*
(5) $\mathcal{F}$ is either a subgroup or a lateral of a subgroup of the multiplicative group $\mathcal{D}$

Proof. Most of the equivalences are either well known or proved in the cited literature. We prove the equivalence of (4). The ideal of a regular design is generated by the basis of the full design and by generating polynomials of the form $X^\alpha - e_\alpha$, where $e_\alpha = \pm 1$; all these polynomials are binomials. Viceversa, if the variety of a binomial ideal is a fraction of $\mathcal{D}$, then all the polynomials $x_i^2 - 1$ are contained in its ideal, and every other binomial in the basis, say $x^\alpha - ex^\beta$, $e = \pm 1$, is equivalent to the generating polynomial $x^{\alpha+\beta} - e$. $\square$

The equivalence theorem 2.1 will be repeatedly used in the next sections. We discuss now the simplest cases of *1-point* and *2-point* fractions.

We could check that every 1-point fraction is regular by using any of the equivalent characterization. For example, can prove the statement using design ideals. A single generic point is $a = (a_1, \ldots, a_m) \in \mathcal{D}$. A binomial basis is $\{x_i - a_i, i = 1, \ldots, m\}$ and, therefore, $\mathcal{F} \equiv \{a\}$ is regular. Or, we could use indicator functions. Indeed, the indicator function of a single point a is $F_a = \frac{1}{2^m}(1 + a_1 x_1) \cdot \cdots \cdot (1 + a_m x_m)$ and F_a meets the requirements for being an indicator function of a regular design.

It appears to be less trivial to prove that every 2-point fraction is regular. Let $\underline{1} = (1, \cdots, 1)$ be the null element of $\mathcal{D}$. We observe that every subset $\mathcal{F}$ of $\mathcal{D}$ made up of two elements, say a and b with $a \neq b$ is a subgroup or a coset of a subgroup. Indeed if $a = \underline{1}$ or $b = \underline{1}$ then $\mathcal{F}$ is a subgroup. If $a \neq \underline{1}$ and $b \neq \underline{1}$ then $\mathcal{F}$ is the coset aH where H is the subgroup $\{\underline{1}, a^{-1}b\}$. We can also prove the same by comparing the number of 2-point subsets with the number of subgroups of order 2. The number of 2-point fractions of $\mathcal{D}$ is $2^{m-1} \cdot (2^m - 1)$. On the other end, every regular fraction is a subgroup of $\mathcal{D}$ or a coset of a subgroup of $\mathcal{D}$. In particular the number of regular fractions of size 2 is equivalent to the number of subgroups of order 2 multiplied by the number of cosets of a subgroup, that is 2^{m-1}. The number of subgroups of order equal to 2 is $2^m - 1$. Indeed every set $\{\underline{1}, p\}$ with $\underline{1} = (1, \cdots, 1)$ and $p \in \mathcal{D}, p \neq \underline{1}$ is a subgroup of order equal to 2.

It follows that the number of regular fractions of size 2 will be equal to $2^{m-1} \cdot (2^m - 1)$, which is the number of 2-point fraction.

If we consider 2^k-point fractions ($k \geq 2$) a similar argument is not valid as it will be clear in the next sections.

3 Union of regular designs

In this section we consider the union of regular designs. To simplify formulæ we will introduce the following notation:

$$X^{\alpha} \equiv X_1^{\alpha_1} \cdot \dots \cdot X_m^{\alpha_m} = X_{\bar{\alpha}}$$

where $\bar{\alpha}$ is the support of α, i.e. the set of indices for which $\alpha_i \neq 0$. We will also write α in place of $\bar{\alpha}$ with a small abuse of notation. We will indicate with $|\alpha|$ the order of the monomial. As an example let's consider $m = 4$ and $\alpha = (0, 1, 1, 0)$. It follows that $X^{\alpha} = X_2 X_3$ will be written as X_{23}. In this case $|\alpha| = 2$.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two regular fraction of $\mathcal{D}$. The indicator functions of $\mathcal{F}_1$ and $\mathcal{F}_2$, say F_1 and F_2 respectively, allow to easily compute the indicator function of the union of $\mathcal{F}_1$ and $\mathcal{F}_2$, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ as $F = F_1 + F_2 - F_1 \times F_2$.

In general, the union of two (disjoint) regular fractions is not a regular fraction. As an example let's consider $m = 2$ factors, $\mathcal{D} = \{-1, +1\} \times \{-1, +1\}$ and $\mathcal{F}_1 = \{(-1, -1)\}$ and $\mathcal{F}_2 = \{(-1, +1), (+1, -1))\}$. Both $\mathcal{F}_1$ and $\mathcal{F}_2$ are regular fractions, according to the propositions of the previous sections. Indeed their indicator functions meet the requirements for regular fractions: $F_1 = \frac{1}{4}(1 - X_1) \cdot (1 - X_2)$ and $F_2 = \frac{1}{2}(1 - X_{12})$. However, the union $\mathcal{F} = \{(-1, -1), (-1, +1), (+1, -1))\}$, is not a regular fraction, because its indicator function is $F = \frac{3}{4} - \frac{1}{4}X_1 - \frac{1}{4}X_2 - \frac{1}{4}X_{12}$.

The same conclusion can be obtained considering *design ideals* related to fractional designs. Given $\mathcal{F}_1 \subset \mathcal{D}$, $\mathcal{F}_2 \subset \mathcal{D}$ and $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ the associated ideals will be $\mathrm{Ideal}\,(\mathcal{F}_1)$, $\mathrm{Ideal}\,(\mathcal{F}_2)$ and $\mathrm{Ideal}\,(\mathcal{F})$. In general, the fact that $\mathrm{Ideal}\,(\mathcal{F}_1)$ and $\mathrm{Ideal}\,(\mathcal{F}_2)$ are binomial ideals by Proposition 2.1 does not imply that $\mathrm{Ideal}\,(\mathcal{F}))$ is a binomial ideal . Indeed, in the previous example the Gröbner bases B_1, B_2 and B of $\mathrm{Ideal}\,(\mathcal{F}_1)$, $\mathrm{Ideal}\,(\mathcal{F}_2)$ and $\mathrm{Ideal}\,(\mathcal{F})$ respectively, are:

$$\begin{aligned} B_1 &= \{X_1 + 1, X_2 + 1\} \\ B_2 &= \left\{X_2^2 - 1, X_1 + X_2\right\} \\ B &= \left\{-1/4X_1X_2 - 1/4X_1 - 1/4X_2 - 1/4, X_2^2 - 1, X_1^2 - 1\right\} \end{aligned}$$

The ideals $\mathrm{Ideal}\,(\mathcal{F}_1)$ and $\mathrm{Ideal}\,(\mathcal{F}_2)$ are both binomial, while $\mathrm{Ideal}\,(\mathcal{F})$ is not.

4 Decomposing a fraction into regular fractions

In this section we want to explore the inverse path, i.e. to analyze the decomposition of a given $\mathcal{F} \subset \mathcal{D}$ into the union of disjoint regular fractions. The first question is: under which condition a regular fraction is a subset of a given fraction?

Theorem 4.1 *Let* $F = \sum_\alpha b_\alpha X^\alpha$ *be the indicator polynomial of a generic fraction* $\mathcal{F} \subset \mathcal{D}$*. Let* $R = \frac{1}{l}\sum_{\alpha\in\mathcal{L}} e_\alpha X^\alpha$ *be the indicator polynomial of a regular fraction* $\mathcal{R} \subset \mathcal{D}$*,* $R = \frac{1}{2^k}(1 + e_1 X^{\alpha_1}) \cdot \ldots \cdot (1 + e_k X^{\alpha_k})$*. The equivalence below follows.*

$$\mathcal{R} \subseteq \mathcal{F} \Leftrightarrow b_0 + e_1 b_{\alpha_1} + \cdots + e_1 \cdot \ldots \cdot e_k b_{\alpha_1+\cdots+\alpha_k} = 1$$

Proof. The set $\mathcal{R}$ is a subset of the set $\mathcal{F}$ if, and only if, the number of points of $\mathcal{R}$ is equal to the number of points of $\mathcal{R}\cap\mathcal{F}$. Therefore, in terms of indicator functions, the equality $\mathcal{R} = \mathcal{R}\cap\mathcal{F}$ is equivalent to $\sum_{t\in\mathcal{D}} F(t)R(t) = \sum_{t\in\mathcal{D}} R(t)$. As

$$\begin{aligned} FR &= (\sum_\alpha b_\alpha X^\alpha) \cdot \frac{1}{2^k}(1 + e_1 X^{\alpha_1}) \cdot \ldots \cdot (1 + e_k X^{\alpha_k}) \\ &= \frac{1}{2^k}\sum_\alpha b_\alpha X^\alpha + \frac{1}{2^k}\sum_\alpha b_\alpha X^\alpha e_1 X^{\alpha_1} + \ldots + \frac{1}{2^k}\sum_\alpha b_\alpha X^\alpha e_1 \cdot \ldots \cdot e_k X^{\alpha_1+\ldots+\alpha_k}, \end{aligned}$$

it follows that

$$\sum_{t\in\mathcal{D}} F(t)R(t) = \frac{1}{2^k}2^m b_0 + \frac{1}{2^k}2^m e_1 b_{\alpha_1} + \ldots + \frac{1}{2^k}2^m e_1 \cdot \ldots \cdot e_k b_{\alpha_1+\ldots+\alpha_k}$$

This quantity is equal to $\sum_{t\in\mathcal{D}} R(t) = \frac{1}{2^k}2^m$ if, and only if,

$$b_0 + e_1 b_{\alpha_1} + \ldots + e_1 \cdot \ldots \cdot e_k b_{\alpha_1+\ldots+\alpha_k} = 1$$

□

Corollary 4.1.1 *A necessary, but not sufficient, condition for a regular fraction* $\mathcal{R}$ *to be contained in* $\mathcal{F}$ *is*

$$b_0 + |b_{\alpha_1}| + \cdots + |b_{\alpha_1+\cdots+\alpha_k}| \geq 1$$

Remark 4.1 *The proof of Theorem 4.1 is based on the fact that a set* $\mathcal{R}$ *is a subset of the set* $\mathcal{F}$ *if, and only if, the number of points of* $\mathcal{R}$ *is equal to the number of points of* $\mathcal{R}\cap\mathcal{F}$*. We can use a similar idea to search for the smallest regular fraction that contains a given design. This problem has already been solved in Pistone and Rogantin (2008). Here we will provide a different proof. Indeed, it is enough to consider the equality* $\mathcal{R}\cap\mathcal{F} = \mathcal{F}$ *that is equivalent to* $\sum_{t\in\mathcal{D}} F(t)R(t) = \sum_{t\in\mathcal{D}} F(t)$*. We get*

$$\frac{1}{2^k}2^m b_0 + \frac{1}{2^k}2^m e_1 b_{\alpha_1} + \ldots + \frac{1}{2^k}2^m e_1 \cdot \ldots \cdot e_k b_{\alpha_1+\ldots+\alpha_k} = 2^m b_0$$

or, equivalently,

$$1 + e_1 \frac{b_{\alpha_1}}{b_0} + \ldots + e_1 \cdot \ldots \cdot e_k \frac{b_{\alpha_1 + \ldots + \alpha_k}}{b_0} = 2^k$$

The left-hand side has 2^k terms and we know, from Pistone and Rogantin (2008), the expression of the coefficients of F

$$b_\alpha = \frac{1}{2^m} \sum_{t \in \mathcal{F}} X^\alpha(t)$$

It follows that $b_\alpha \leq b_0$ and, therefore, the previous equality will be satisfied if, and only if, each term will be equal to 1 that is

$$\begin{cases} \frac{b_{\alpha_1}}{b_0} = e_1 \\ \ldots \\ \frac{b_{\alpha_1 + \ldots + \alpha_k}}{b_0} = e_1 \cdot \ldots \cdot e_k \end{cases}$$

The set $\mathcal{L}$ of α such that $\frac{b_\alpha}{b_0} = e_\alpha$ is a subgroup of L. First we observe that if we consider $\alpha \in \mathcal{L}$, we get

$$b_\alpha = e_\alpha b_0 = e_\alpha \frac{1}{2^m} \sum_{t \in \mathcal{F}} X^0(t) = e_\alpha \frac{1}{2^m} \# (\mathcal{F}) = \frac{1}{2^m} \sum_{t \in \mathcal{F}} X^\alpha(t)$$

where $\# (\mathcal{F})$ is the number of points of $\mathcal{F}$ and therefore

$$e_\alpha \# (\mathcal{F}) = \sum_{t \in \mathcal{F}} X^\alpha(t)$$

It follows that it must be $X^\alpha(t) = e_\alpha$ for all $t \in \mathcal{F}$. We can nov prove that $\mathcal{L}$ is a subgroup of L:

- *$\alpha = 0 \in \mathcal{L}$, being $\frac{b_0}{b_0} = 1$;*
- *given $\alpha \in \mathcal{L}$, α itself is the inverse of α;*
- *given $\alpha_1, \alpha_2 \in \mathcal{L}$, we get $\alpha_1 + \alpha_2 \in \mathcal{L}$. Indeed*

$$\frac{b_{\alpha_1 + \alpha_2}}{b_0} = \frac{\frac{\sum_{t \in \mathcal{F}} X^{\alpha_1}(t) X^{\alpha_2}(t)}{2^m}}{\frac{\#(\mathcal{F})}{2^m}} = \frac{\# (\mathcal{F}) \, e_{\alpha_1} e_{\alpha_2}}{\# (\mathcal{F})} = e_{\alpha_1} e_{\alpha_2}$$

It follows that the smallest regular fraction containing $\mathcal{F}$ is

$$\frac{1}{2^k} \sum_{\alpha \in \mathcal{L}} X^\alpha$$

4.1 A small example

Let's consider the 3-point fraction $\mathcal{F} \subset \mathcal{D} = \{-1,+1\}^2$ we have already considered in the previous section, $\mathcal{F} = \{(-1,-1),(-1,+1),(+1,-1))\}$. The indicator function F of $\mathcal{F}$ is $F = \frac{3}{4} - \frac{1}{4}X_1 - \frac{1}{4}X_2 - \frac{1}{4}X_1X_2$, that is $b_0 = \frac{3}{4}, b_1 = -\frac{1}{4}, b_2 = -\frac{1}{4}, b_{12} = -\frac{1}{4}$. The following equations are all true

$$\begin{cases} b_0 - b_1 = 1 \\ b_0 - b_2 = 1 \\ b_0 - b_{12} = 1 \\ b_0 - b_1 - b_2 + b_{12} = 1 \\ b_0 - b_1 + b_2 - b_{12} = 1 \\ b_0 + b_1 - b_2 - b_{12} = 1 \end{cases}$$

From each equation, using theorem 4.1, we obtain a regular fraction that is a subset of $\mathcal{F}$.

Indicator function	Regular fraction
$F_1 = \frac{1}{2}(1 - X_1)$	$\mathcal{F}_1 = \{(-1,-1),(-1,+1)\}$
$F_2 = \frac{1}{2}(1 - X_2)$	$\mathcal{F}_2 = \{(-1,-1),(+1,-1)\}$
$F_3 = \frac{1}{2}(1 - X_{12})$	$\mathcal{F}_3 = \{(-1,+1),(+1,-1)\}$
$F_4 = \frac{1}{4}(1 - X_1)(1 - X_2)$	$\mathcal{F}_4 = \{(-1,-1)\}$
$F_5 = \frac{1}{4}(1 - X_1)(1 + X_2)$	$\mathcal{F}_5 = \{(-1,+1)\}$
$F_6 = \frac{1}{4}(1 + X_1)(1 - X_2)$	$\mathcal{F}_6 = \{(+1,-1)\}$

5 Plackett-Burman designs

More interesting examples are obtained considering Plackett-Burman designs (Plackett and Burman, 1946). In particular, the Plackett-Burman design for 11 variables and 12 runs is built according to the following procedure:

(1) the first row, called key, is given as: $++-+++---+-$;
(2) the second row up to the eleventh row are built right-shifting the key of one position each time;
(3) the 12th row is set equal to $-----------$.

The Plackett-Burman array for eleven factors is listed below. We consider the case with $m = 5$ factors and, from the Plackett-Burman array for 11 factors we have randomly selected the following fraction $\mathcal{F}$, which corresponds to columns A, B, F, H, I. The plus sign '+' has been coded with '1' and the minus sign '−' with '−1'.

N	A	B	C	D	E	F	G	H	I	J	K
1	+	+	−	+	+	+	−	−	−	+	−
2	−	+	+	−	+	+	+	−	−	−	+
3	+	−	+	+	−	+	+	+	−	−	−
4	−	+	−	+	+	−	+	+	+	−	−
5	−	−	+	−	+	+	−	+	+	+	−
6	−	−	−	+	−	+	+	−	+	+	+
7	+	−	−	−	+	−	+	+	−	+	+
8	+	+	−	−	−	+	−	+	+	−	+
9	+	+	+	−	−	−	+	−	+	+	−
10	−	+	+	+	−	−	−	+	−	+	+
11	+	−	+	+	+	−	−	−	+	−	+
12	−	−	−	−	−	−	−	−	−	−	−

$\mathcal{F} =$

N	X_1	X_2	X_3	X_4	X_5
1	1	1	1	1	1
2	1	1	−1	−1	1
3	1	−1	−1	−1	1
4	−1	1	−1	1	1
5	−1	−1	1	1	1
6	−1	−1	1	−1	1
7	1	1	1	−1	−1
8	1	−1	1	1	−1
9	1	−1	−1	1	−1
10	−1	1	1	−1	−1
11	−1	1	−1	1	−1
12	−1	−1	−1	−1	−1

The indicator function of $\mathcal{F}$ is

$$
\begin{aligned}
F = \frac{3}{8} + \frac{1}{8}X_{345} + \frac{1}{8}X_{245} - \frac{1}{8}X_{235} - \frac{1}{8}X_{234} + \\
\frac{1}{8}X_{2345} - \frac{1}{8}X_{145} - \frac{1}{8}X_{135} + \frac{1}{8}X_{134} + \frac{1}{8}X_{1345} + \frac{1}{8}X_{125} + \\
- \frac{1}{8}X_{124} + \frac{1}{8}X_{1245} + \frac{1}{8}X_{123} + \frac{1}{8}X_{1235} + \frac{1}{8}X_{1234}
\end{aligned}
$$

In particular, it follows from the inspection of the b_α's that it is not a regular fraction.

Now we proceed to search for regular fractions that are contained in $\mathcal{F}$. The first constraint concerns the size of the regular fraction. It must be less or equal to 12, the number of points of $\mathcal{F}$. Being $\mathcal{R}$ a regular fraction, it follows that the size of $\mathcal{R}$ could be $2^0 = 1$ or $2^1 = 2$ or $2^2 = 4$ or $2^3 = 8$. We already know, from Section 2 that each of the 12 points of $\mathcal{F}$ are 1-point regular fraction and each of the $\binom{12}{2} = 66$ 2-point subsets are regular fractions. Let us study 4-point and 8-point subsets of $\mathcal{F}$.

The corollary of Theorem 4.1 allows us to exclude the existence of 8-point regular fractions. Indeed, the following condition should be true for a proper choice of e_1, e_2 and α_1, α_2

$$b_0 + e_1 b_{\alpha_1} + e_2 b_{\alpha_2} + e_1 e_2 b_{\alpha_1+\alpha_2} = 1$$

As $b_0 = \frac{3}{8}$ and the absolute value of b_i is $\frac{1}{8}$, for all i's, it is not possible that the left-hand side of the previous equation sums up to 1.

Finally, we investigate 4-point regular fractions. For a proper choice of e_1, e_2, e_3 and $\alpha_1, \alpha_2, \alpha_3$ the following equation should hold true

$$b_0 + e_1 b_{\alpha_1} + e_2 b_{\alpha_2} + e_3 b_{\alpha_3} + e_1 e_2 b_{\alpha_1+\alpha_2} + e_1 e_3 b_{\alpha_1+\alpha_3} + e_2 e_3 b_{\alpha_2+\alpha_3} + \\ e_1 e_2 e_3 b_{\alpha_1+\alpha_2+\alpha_3} = 1$$

A subgroup of order eight will be $\{\underline{1}, a, b, ab, c, ac, bc, abc\}$ with $a \neq \underline{1}$, $b \neq \underline{1}$, $c \neq \underline{1}$ and $a \neq b$, $a \neq c$ and $b \neq c$. We can choose a, b and c in $\binom{31}{2}(31-3)$ different ways. The number of different subgroups is obtained dividing this number by $\binom{7}{2}4$. We get 155 different subgroups.

Every subgroup of order 8, $\mathcal{S}_i^{(8)} =< \alpha_{1i}, \alpha_{2i}, \alpha_{3i} >, i = 1, \ldots, 155$ defines 8 regular fractions of size 4 (the subgroup orthogonal to $\mathcal{S}_i^{(8)}$ and its cosets). To find the regular fractions embedded into $\mathcal{F}$ we must solve the following systems of equations ($i = 1, \ldots, 155$)

$$\begin{cases} e_1^2 - 1 = 0 \\ e_2^2 - 1 = 0 \\ e_3^2 - 1 = 0 \\ b_0 + e_1 b_{\alpha_{1i}} + e_2 b_{\alpha_{2i}} + e_3 b_{\alpha_{3i}} + e_1 e_2 b_{\alpha_{1i}+\alpha_{2i}} + \\ \quad e_1 e_3 b_{\alpha_{1i}+\alpha_{3i}} + e_2 e_3 b_{\alpha_{2i}+\alpha_{3i}} + e_1 e_2 e_3 b_{\alpha_{1i}+\alpha_{2i}+\alpha_{3i}} - 1 = 0 \end{cases} \tag{2}$$

To do it we generate the 155 subgroups of $\mathcal{D}$ of order eight (for example using GAP (2007)) and then we search for the solution of the system, if any, simply checking all the eight possible triples (e_1, e_2, e_3). Let's consider $\mathcal{S}_1 =< \{1\}, \{2\}, \{3\} >$. Being $b_0 = \frac{3}{8}$, $b_1 = b_2 = b_3 = b_{12} = b_{13} = b_{23} = 0$ and $b_{123} = \frac{1}{8}$ the corresponding system of equation is

$$\begin{cases} e_1^2 - 1 = 0 \\ e_2^2 - 1 = 0 \\ e_3^2 - 1 = 0 \\ \frac{3}{8} + \frac{1}{8} e_1 e_2 e_3 - 1 = 0 \end{cases}$$

The system doesn't have any solution.

Let's now consider another subgroup, for example $\mathcal{S}_2 =< \{4\}, \{12\}, \{135\} >$. Being $b_0 = \frac{3}{8}$, $b_4 = b_{12} = 0$, $b_{135} = b_{124} = b_{235} = -\frac{1}{8}$ and $b_{1345} = b_{2345} = \frac{1}{8}$ the corresponding system of equation is

$$\begin{cases} e_1^2 - 1 = 0 \\ e_2^2 - 1 = 0 \\ e_3^2 - 1 = 0 \\ \frac{3}{8} - \frac{1}{8}e_3 - \frac{1}{8}e_1e_2 + \frac{1}{8}e_1e_3 - \frac{1}{8}e_2e_3 + \frac{1}{8}e_1e_2e_3 - 1 = 0 \end{cases}$$

The solution is $e_1 = -1, e_2 = 1, e_3 = -1$. It defines the indicator function $F^{(1)} = \frac{1}{8}(1 - X_4)(1 + X_{12})(1 - X_{135})$. The corresponding set of points $\mathcal{F}^{(1)}$ is

N	X_1	X_2	X_3	X_4	X_5
2	1	1	−1	−1	1
6	−1	−1	1	−1	1
7	1	1	1	−1	−1
12	−1	−1	−1	−1	−1

To proceed into the decomposition of $\mathcal{F}$ we remove from $\mathcal{F}$ itself the points of $\mathcal{F}^{(1)}$. The indicator function of the new set will be $F - F^{(1)}$:

$$\frac{1}{4} + \frac{1}{8}X_4 - \frac{1}{8}X_{12} + \frac{1}{8}X_{345} + \frac{1}{8}X_{245} - \frac{1}{8}X_{234} + \\ - \frac{1}{8}X_{145} + \frac{1}{8}X_{134} + \frac{1}{8}X_{125} + \frac{1}{8}X_{1245} + \frac{1}{8}X_{123} + \\ + \frac{1}{8}X_{1235} + \frac{1}{8}X_{1234}$$

We now search for regular fractions contained into $\mathcal{F} - \mathcal{F}^{(1)}$. A regular fraction $\mathcal{R}$, in order to be contained into $\mathcal{F} - \mathcal{F}^{(1)}$, must be contained into $\mathcal{F}$. We can therefore limit our search to the solutions that we have identified in the first step. Let's now consider $\mathcal{S}_3 =< \{12\}, \{35\}, \{245\} >$.

Being $b_0^{(1)} = \frac{1}{4}$, $b_{35}^{(1)} = 0$ $b_{245}^{(1)} = b_{134}^{(1)} = b_{1235}^{(1)} = \frac{1}{8}$ and $b_{234}^{(1)} = b_{145}^{(1)} = b_{12}^{(1)} = -\frac{1}{8}$ the corresponding system of equation is

$$\begin{cases} e_1^2 - 1 = 0 \\ e_2^2 - 1 = 0 \\ e_3^2 - 1 = 0 \\ \frac{1}{4} - \frac{1}{8}e_1 + \frac{1}{8}e_3 + \frac{1}{8}e_1e_2 - \frac{1}{8}e_1e_3 - \frac{1}{8}e_2e_3 + \frac{1}{8}e_1e_2e_3 - 1 = 0 \end{cases}$$

The solution is $e_1 = -1, e_2 = -1, e_3 = 1$. It defines the indicator function $F^{(2)} = \frac{1}{8}(1 - X_{12})(1 - X_{35})(1 + X_{245})$. The corresponding set of points $\mathcal{F}^{(2)}$ is

N	X_1	X_2	X_3	X_4	X_5
3	1	−1	−1	−1	1
4	−1	1	−1	1	1
8	1	−1	1	1	−1
10	−1	1	1	−1	−1

If we remove this set of points from $\mathcal{F} - \mathcal{F}_1$ we get the following indicator function $F^{(3)} = F - F^{(1)} - F^{(2)}$:

$$\frac{1}{8} + \frac{1}{8}X_4 + \frac{1}{8}X_{35} + \frac{1}{8}X_{345} + \frac{1}{8}X_{125} + \frac{1}{8}X_{1245} + \frac{1}{8}X_{123} + \frac{1}{8}X_{1234}$$

or, equivalently,

$$\frac{1}{8}(1 + X_4)(1 + X_{35})(1 + X_{125})$$

The corresponding set of points $\mathcal{F}^{(3)}$ is

N	X_1	X_2	X_3	X_4	X_5
1	1	1	1	1	1
5	−1	−1	1	1	1
9	1	−1	−1	1	−1
11	−1	1	−1	1	−1

$F^{(3)}$ meets the requirements to be an indicator function of a regular design. We have therefore decomposed $\mathcal{F}$ into the union of three disjoint regular designs, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.

5.1 *Finding all possible decomposition of the given Plackett-Burman design into the union of 4-point regular designs*

In this part we find all the possible decompositions of the given "Plackett-Burman" design. As described in the previous section, we consider all the 155 subgroups of order 8, $\mathcal{S}_i^{(8)} =< \alpha_{1i}, \alpha_{2i}, \alpha_{3i} >, i = 1, \ldots, 155$ and we search for the solutions of the systems of equations (2).

We find that 15 of these 155 systems of equations have a non-empty set of solutions. Each of these non-empty set has its indicator function $R_j, j = 1, \ldots, 15$:

$$
\begin{aligned}
R_1 &= \tfrac{1}{8}(1 - X_4)(1 + X_{12})(1 - X_{235}) \\
R_2 &= \tfrac{1}{8}(1 + X_1)(1 + X_{23})(1 + X_{245}) \\
R_3 &= \tfrac{1}{8}(1 + X_1)(1 - X_{45})(1 - X_{235}) \\
R_4 &= \tfrac{1}{8}(1 - X_2)(1 + X_{34})(1 - X_{145}) \\
R_5 &= \tfrac{1}{8}(1 + X_2)(1 + X_{15})(1 - X_{345}) \\
R_6 &= \tfrac{1}{8}(1 - X_{23})(1 - X_{45})(1 - X_{135}) \\
R_7 &= \tfrac{1}{8}(1 - X_3)(1 + X_{25})(1 - X_{145}) \\
R_8 &= \tfrac{1}{8}(1 + X_3)(1 + X_{14})(1 - X_{245}) \\
R_9 &= \tfrac{1}{8}(1 - X_{14})(1 - X_{25})(1 + X_{345}) \\
R_{10} &= \tfrac{1}{8}(1 - X_{15})(1 - X_{34})(1 + X_{245}) \\
R_{11} &= \tfrac{1}{8}(1 - X_5)(1 + X_{13})(1 - X_{234}) \\
R_{12} &= \tfrac{1}{8}(1 + X_4)(1 + X_{35})(1 - X_{125}) \\
R_{13} &= \tfrac{1}{8}(1 + X_5)(1 + X_{24})(1 - X_{134}) \\
R_{14} &= \tfrac{1}{8}(1 - X_{12})(1 - X_{35})(1 + X_{245}) \\
R_{15} &= \tfrac{1}{8}(1 - X_{13})(1 - X_{24})(1 + X_{345})
\end{aligned}
$$

To build a decomposition of $\mathcal{F}$, we start from one of these indicator function, let's say R_1 (that identifies the regular fraction $\mathcal{R}_1$). We have now to choose another indicator function R_k, in the set $R_2, \ldots, R_{15}$, with the condition that the corresponding regular fraction $\mathcal{R}_k$ doesn't intersect $\mathcal{R}_1$: $\mathcal{R}_1 \cap \mathcal{R}_k = \emptyset$. We have two possible choices, R_{12} and R_{14}. If we choose R_{12} we have to take R_{14} to complete the decomposition and, viceversa, if we choose R_{14} we have to take R_{12} to complete the decomposition. Repeating the same procedure for all the $R_j, j = 2, \ldots, 15$ and considering only the different decompositions, we get that $\mathcal{F}$ can be considered as the following union of three, mutually disjoint, regular 4-point designs

$$
\begin{aligned}
\mathcal{F} &= \mathcal{R}_1 \cup \mathcal{R}_{12} \cup \mathcal{R}_{14} \\
\mathcal{F} &= \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_6 \\
\mathcal{F} &= \mathcal{R}_4 \cup \mathcal{R}_5 \cup \mathcal{R}_{10} \\
\mathcal{F} &= \mathcal{R}_7 \cup \mathcal{R}_8 \cup \mathcal{R}_9 \\
\mathcal{F} &= \mathcal{R}_{11} \cup \mathcal{R}_{13} \cup \mathcal{R}_{15}
\end{aligned}
$$

We observe that the decomposition that has been found in the previous section is the first one, $\mathcal{F} = \mathcal{R}_1 \cup \mathcal{R}_{12} \cup \mathcal{R}_{14}$.

5.2 *Decomposing all the Plackett-Burman designs with m=5 and 12 different runs into the unions of 4-point regular designs*

Using an ad-hoc software routine written in SAS IML, we consider all the $\binom{11}{5} = 462$ different designs that can be obtained choosing 5 columns out of the 11 of the original designs. We classify them into classes made by equal designs. We get the following table where the first column contains an identification of the design that can be chosen as representative of the class, the second column reports the number of designs that are contained in the class and the third column gives the number of different runs contained in the designs of the class. For example, the design $\mathcal{F}$ that we have considered in the previous sections, belongs to the class whose representative is the design number "69". There are 11 designs that are equal to $\mathcal{F}$ and each one has 12 points.

ID	*N*	*SIZE*	*ID*	*N*	*SIZE*	*ID*	*N*	*SIZE*
1	8	12	28	6	12	72	5	11
2	7	12	30	10	12	73	6	12
3	6	12	32	6	11	74	5	12
4	8	12	35	6	12	82	6	12
5	5	12	37	3	12	84	2	12
6	7	11	39	4	12	85	9	11
7	2	12	44	11	12	87	7	12
8	13	12	45	7	12	89	4	12
9	6	12	46	6	12	94	6	12
10	11	11	49	2	12	98	7	12
11	7	12	51	7	12	100	3	12
12	7	12	52	9	12	101	8	12
13	5	12	53	5	12	102	3	11
14	7	11	54	4	11	103	7	12
15	10	12	55	4	12	110	2	12
16	6	12	57	3	11	116	5	12
17	7	12	58	6	12	117	1	12
18	3	12	61	6	12	128	2	12
19	7	12	63	4	12	134	5	12
20	11	12	64	3	12	140	3	12
21	5	12	65	8	12	146	5	11
22	8	12	66	5	12	147	3	12
23	4	12	67	2	12	149	4	12
24	7	12	68	7	12	154	6	12
25	2	12	69	11	12	159	1	12
26	5	12	70	13	12	167	2	12
27	6	11	71	6	12	184	1	12

It follows that the 462 designs can be partitioned into 81 classes:

- 70 classes where each design contains 12 runs;
- 11 classes where each design contains 11 runs.

In this section we analyse the designs with 12 different runs, in the next one the designs with 11 different runs. To do that we repeat the procedure described in the previous section for all the 70 different 12-run designs. So, first we determine the indicator functions of all the 70 designs. We observe that each indicator function has the following form:

$$\begin{aligned}
&\frac{3}{8} + a_{345}X_{345} + a_{245}X_{245} + a_{235}X_{235} + a_{234}X_{234} + \\
&\quad a_{2345}X_{2345} + a_{145}X_{145} + a_{135}X_{135} + a_{134}X_{134} + a_{1345}X_{1345} + a_{125}X_{125} + \\
&\qquad a_{124}X_{124} + a_{1245}X_{1245} + a_{123}X_{123} + a_{1235}X_{1235} + a_{1234}X_{1234}
\end{aligned}$$

where the coefficients $a_{345}, \ldots, a_{1234}$ are all equal to $\pm\frac{1}{8}$.

Then we decompose each fraction into three disjoint 4-point regular designs. As for the design considered in the previous section we obtain that every design

- contains 15 "4-point regular design"
- can be considered as the union of three regular designs in 5 different ways

At this point we examine the structure of the decompositions of all the 70 designs. For all the decompositions, if we indicate with R_1, R_2 and R_3 the indicator functions involved, we get

$$\begin{aligned}
R_1 &= \tfrac{1}{8}(1 + e_1X_{\alpha_1} + e_2X_{\alpha_2} + e_1e_2X_{\alpha_1+\alpha_2} + e_4X_{\alpha_4} + e_1e_4X_{\alpha_1+\alpha_4} \\
&\qquad + e_2e_4X_{\alpha_2+\alpha_4} + e_1e_2e_4X_{\alpha_1+\alpha_2+\alpha_4}) \\
R_2 &= \tfrac{1}{8}(1 - e_1X_{\alpha_1} + e_3X_{\alpha_3} - e_1e_3X_{\alpha_1+\alpha_3} + e_5X_{\alpha_5} - e_1e_5X_{\alpha_1+\alpha_5} \\
&\qquad + e_2e_5X_{\alpha_2+\alpha_5} + e_1e_3e_5X_{\alpha_1+\alpha_3+\alpha_5}) \\
R_3 &= \tfrac{1}{8}(1 - e_2X_{\alpha_2} - e_3X_{\alpha_3} + e_2e_3X_{\alpha_2+\alpha_3} + e_6X_{\alpha_6} - e_2e_6X_{\alpha_2+\alpha_6} \\
&\qquad - e_3e_6X_{\alpha_3+\alpha_6} + e_2e_3e_6X_{\alpha_2+\alpha_3+\alpha_6})
\end{aligned}$$

where

- $|\alpha_1|$, $|\alpha_2|$ and $|\alpha_3|$ are less than three;
- all the others, i.e. $|\alpha_1+\alpha_2|$, ..., $|\alpha_2+\alpha_3+\alpha_6|$ are greater or equal to 3.

This evidence has suggested us the following 2-step procedure.

(1) We generate all the $\alpha_1, \ldots, \alpha_6$ that satisfy the previous requirements, i.e. $|\alpha_1|$, $|\alpha_2|$ and $|\alpha_3|$ less than three and $|\alpha_1+\alpha_2|$, ..., $|\alpha_2+\alpha_3+\alpha_6|$ greater

or equal to 3:

N	α_1	α_2	α_3	α_4	α_5	α_6
1	1	23	45	245	234	124
2	1	24	35	235	234	123
3	1	25	34	234	235	123
4	2	13	45	145	134	124
5	2	14	35	135	134	123
6	2	15	34	134	135	123
7	3	12	45	145	124	134
8	3	14	25	125	124	123
9	3	15	24	124	125	123
10	4	12	35	135	123	134
11	4	13	25	125	123	124
12	4	15	23	123	125	124
13	5	12	34	134	123	135
14	5	13	24	124	123	125
15	5	14	23	123	124	125

(2) For every choice of $\alpha_1, \ldots, \alpha_6$ we build the 64 indicator functions that correspond to all the values of $e_1, \ldots, e_6$, being $e_i = \pm 1, i = 1, \ldots, 6$.

According to this procedure we have generated $15 \times 64 = 960$ indicator functions. If we limit to the different ones we get *192 indicator functions*. This number is the same that has been found in Carlini and Pistone (2007), as the total number of orthogonal arrays with 12 runs and strength 2. It is interesting to point out that the understanding of the mechanism underlying the Plackett-Burman designs (m=5, 12 runs) has allowed to build *all* the orthogonal arrays of strength 2.

5.3 Decomposing all the Plackett-Burman designs with m=5 and 11 different runs into the union of regular designs

We repeat the decomposition process for the 11-run designs. As reported in the previous section there are 11 classes of different 11-run designs. It results that every design contains 15 different 4-run regular designs of the following form

$$
\begin{aligned}
R_1 &= \tfrac{1}{8}(1+e_1X_1)(1+e_{23}X_{23})(1+e_{45}X_{45})\\
R_2 &= \tfrac{1}{8}(1+e_1X_1)(1+e_{24}X_{24})(1+e_{35}X_{35})\\
R_3 &= \tfrac{1}{8}(1+e_1X_1)(1+e_{25}X_{25})(1+e_{34}X_{34})\\
R_4 &= \tfrac{1}{8}(1+e_2X_2)(1+e_{13}X_{13})(1+e_{45}X_{45})\\
R_5 &= \tfrac{1}{8}(1+e_2X_2)(1+e_{14}X_{14})(1+e_{35}X_{35})\\
R_6 &= \tfrac{1}{8}(1+e_2X_2)(1+e_{15}X_{15})(1+e_{34}X_{34})\\
R_7 &= \tfrac{1}{8}(1+e_3X_3)(1+e_{12}X_{12})(1+e_{45}X_{45})\\
R_8 &= \tfrac{1}{8}(1+e_3X_3)(1+e_{14}X_{14})(1+e_{25}X_{25})\\
R_9 &= \tfrac{1}{8}(1+e_3X_3)(1+e_{15}X_{15})(1+e_{24}X_{24})\\
R_{10} &= \tfrac{1}{8}(1+e_4X_4)(1+e_{12}X_{12})(1+e_{35}X_{35})\\
R_{11} &= \tfrac{1}{8}(1+e_4X_4)(1+e_{13}X_{13})(1+e_{25}X_{25})\\
R_{12} &= \tfrac{1}{8}(1+e_4X_4)(1+e_{15}X_{15})(1+e_{23}X_{23})\\
R_{13} &= \tfrac{1}{8}(1+e_5X_5)(1+e_{12}X_{12})(1+e_{34}X_{34})\\
R_{14} &= \tfrac{1}{8}(1+e_5X_5)(1+e_{13}X_{13})(1+e_{24}X_{24})\\
R_{15} &= \tfrac{1}{8}(1+e_5X_5)(1+e_{14}X_{14})(1+e_{23}X_{23})
\end{aligned}
$$

where $e_\alpha = \pm 1$ for all $\alpha \in \{1, 2, \ldots, 5, 12, 13, \ldots, 45\}$ and $R_i \cap R_j \neq \emptyset$ for all $i, j = 1, \ldots, 15,\ i \neq j$.

For example let's consider $\mathcal{D}_1$ as the following 11-run design

N	X_1	X_2	X_3	X_4	X_5	Y_C
1	-1	-1	-1	-1	-1	1
2	-1	-1	-1	1	1	1
3	-1	-1	1	-1	1	1
4	-1	1	-1	1	-1	1
5	-1	1	1	-1	-1	1
6	-1	1	1	1	1	1
7	1	-1	-1	-1	1	1
8	1	-1	1	1	-1	2
9	1	1	-1	-1	-1	1
10	1	1	-1	1	1	1
11	1	1	1	-1	1	1

where Y_C are values of the counting function for all the points of the design (the run 8 is replicated).

The 15 different 4-run regular designs contained into $\mathcal{D}_1$ are:

$$
\begin{aligned}
R_1 &= \tfrac{1}{8}(1-X_1)(1+X_{23})(1+X_{45})\\
R_2 &= \tfrac{1}{8}(1-X_1)(1+X_{24})(1+X_{35})\\
R_3 &= \tfrac{1}{8}(1-X_1)(1-X_{25})(1-X_{34})\\
R_4 &= \tfrac{1}{8}(1+X_2)(1-X_{13})(1+X_{45})\\
R_5 &= \tfrac{1}{8}(1+X_2)(1-X_{14})(1+X_{35})\\
R_6 &= \tfrac{1}{8}(1+X_2)(1+X_{15})(1-X_{34})\\
R_7 &= \tfrac{1}{8}(1-X_3)(1+X_{12})(1+X_{45})\\
R_8 &= \tfrac{1}{8}(1-X_3)(1-X_{14})(1-X_{25})\\
R_9 &= \tfrac{1}{8}(1-X_3)(1+X_{15})(1+X_{24})\\
R_{10} &= \tfrac{1}{8}(1-X_4)(1+X_{12})(1+X_{35})\\
R_{11} &= \tfrac{1}{8}(1-X_4)(1-X_{13})(1-X_{25})\\
R_{12} &= \tfrac{1}{8}(1-X_4)(1+X_{15})(1+X_{23})\\
R_{13} &= \tfrac{1}{8}(1+X_5)(1+X_{12})(1-X_{34})\\
R_{14} &= \tfrac{1}{8}(1+X_5)(1-X_{13})(1+X_{24})\\
R_{15} &= \tfrac{1}{8}(1+X_5)(1-X_{14})(1+X_{23})
\end{aligned}
$$

Being $R_i \cap R_j \neq \emptyset$ for all $i, j = 1, \ldots, 15,\ i \neq j$, it follows that every 11-run design will contain only one 4-run regular design. Because we know that all the 2-point fractions are regular it follows that all the decompositions of all the 11-run Plackett-Burman designs will have the following form:

$$R_a^{(4)} + R_b^{(2)} + R_c^{(2)} + R_d^{(2)} + R_e^{(2)}$$

where $R_a^{(4)}$ is a regular fraction with 4 points and $R_b^{(2)}, R_c^{(2)}, R_d^{(2)}, R_e^{(2)}$ are regular fractions with 2 points each. $R_a^{(4)}, R_b^{(2)}, R_c^{(2)}, R_d^{(2)}, R_e^{(2)}$ must be indicator functions of mutually disjoint regular fractions.

With respect to $\mathcal{D}_1$, let's randomly choose one of the indicator functions corresponding to a 4-run regular fraction contained into it:

$$R_{12} = \frac{1}{8}(1-X_4)(1+X_{15})(1+X_{23})$$

If we denote with $\mathcal{R}_{12}$ the points corresponding to R_{12} we get the following subset of $\mathcal{D}_1$:

N	X_1	X_2	X_3	X_4	X_5	Y_C
1	-1	-1	-1	-1	-1	1
5	-1	1	1	-1	-1	1
7	1	-1	-1	-1	1	1
11	1	1	1	-1	1	1

The subset $\mathcal{R}_{12} \subseteq \mathcal{D}_1$ can now be considered as the disjoint union of 2-run regular fractions. We can choose the following:

N	X_1	X_2	X_3	X_4	X_5	Y_C
2	-1	-1	-1	1	1	1
8	1	-1	1	1	-1	1

N	X_1	X_2	X_3	X_4	X_5	Y_C
3	-1	-1	1	-1	1	1
8	1	-1	1	1	-1	1

N	X_1	X_2	X_3	X_4	X_5	Y_C
4	-1	1	-1	1	-1	1
6	-1	1	1	1	1	1

N	X_1	X_2	X_3	X_4	X_5	Y_C
9	1	1	-1	-1	-1	1
10	1	1	-1	1	1	1

The corresponding indicator functions are:

$$\begin{aligned}
&\tfrac{1}{16}(1+X_4)(1-X_{15})(1-X_2)(1-X_{35})\\
&\tfrac{1}{16}(1-X_{15})(1-X_{23})(1-X_2)(1-X_{45})\\
&\tfrac{1}{16}(1+X_4)(1+X_2)(1+X_{35})(1-X_{14})\\
&\tfrac{1}{16}(1-X_{23})(1+X_2)(1+X_{45})(1+X_1)
\end{aligned}$$

We generalise this result in the following way.

Let's take R, a 4-run regular design whose indicator function is

$$R_a = \frac{1}{8}(1+e_1X_{i_1})(1+e_{23}X_{i_2i_3})(1+e_{45}X_{i_4i_5})$$

where $e_1 = \pm 1$, $e_{23} = \pm 1$, $e_{45} = \pm 1$ and $i_j \in \{1,2,3,4,5\}$, $i_j \neq i_k$ for $j \neq k$

From R_a, considering the form of the 2-points indicator functions that we have just determined (5.3), we build

$$R_b = \tfrac{1}{16}(1 - e_1 X_{i_1})(1 - e_{23} X_{i_2 i_3})(1 + e_4 X_{i_4})(1 + e_{35} X_{i_3 i_5})$$
$$R_c = \tfrac{1}{16}(1 - e_{23} X_{i_2 i_3})(1 - e_{45} X_{i_4 i_5})(1 + e_4 X_{i_4})(1 + e_{13} X_{i_1 i_3})$$
$$R_d = \tfrac{1}{16}(1 - e_1 X_{i_1})(1 - e_4 X_{i_4})(1 - e_{35} X_{i_3 i_5})(1 + e_{13} e_{23} X_{i_1 i_2})$$
$$R_e = \tfrac{1}{16}(1 - e_{45} X_{i_4 i_5})(1 - e_4 X_{i_4})(1 - e_{13} X_{i_1 i_3})(1 + e_{23} e_{35} X_{i_5})$$

where $e_4 = \pm 1$, $e_{35} = \pm 1$, $e_{13} = \pm 1$

If we generate all the monomials $b_\alpha X^\alpha$ of R_a, R_b, R_c, R_d, R_e for which $b_\alpha \neq 0$ and $|\alpha| \leq 2$ we get the following table

$Indicator$	X_{i_1}	$X_{i_2 i_3}$	$X_{i_4 i_5}$	X_{i_4}	$X_{i_3 i_5}$	$X_{i_1 i_3}$	$X_{i_1 i_4}$	$X_{i_2 i_5}$
R_a	$+1/8e_1$	$+1/8e_{23}$	$+1/8e_{45}$					
R_b	$-1/16e_1$	$-1/16e_{23}$		$+1/16e_4$	$+1/16e_{35}$		$-1/16e_1e_4$	$-1/16e_{23}e_{35}$
R_c		$-1/16e_{23}$	$-1/16e_{45}$	$+1/16e_4$		$+1/16e_{13}$		
R_d	$-1/16e_1$			$-1/16e_4$	$-1/16e_{35}$		$+1/16e_1e_4$	
R_e			$-1/16e_{45}$	$-1/16e_4$		$-1/16e_{13}$		$+1/16e_{23}e_{35}$
C	0	0	0	0	0	0	0	0

$Indicator$	X_{i_5}	$X_{i_1 i_2}$	X_{i_2}	$X_{i_2 i_4}$
R_a				
R_b				
R_c	$-1/16e_{45}e_4$	$-1/16e_{13}e_{23}$		
R_d		$+1/16e_{13}e_{23}$	$-1/16e_1e_{13}e_{23}$	$+1/16e_1e_4e_{13}e_{23}$
R_e	$+1/16e_{45}e_4$		$+1/16e_{23}e_{35}e_{45}e_4$	$-1/16e_{45}e_{23}e_{35}$
C	0	0	$-1/16e_{23}(e_1e_{13} - e_{35}e_{45}e_4)$	$+1/16e_{23}(e_1e_4e_{13} - e_{45}e_{35})$

For $C = R_a + R_b + R_c + R_d + R_e$ to be a counting function of an orthogonal array of strength 2, all the b_α for which $|\alpha| = 2$ or $|\alpha| = 3$ must be equal to zero.

It follows that the following condition must be satisfied

$$\begin{cases} -1/16 e_{23}(e_1 e_{13} - e_{35} e_{45} e_4) = 0 \\ +1/16 e_{23}(e_1 e_4 e_{13} - e_{45} e_{35}) = 0 \end{cases}$$

This couple of conditions are equivalent to $e_1 e_4 e_{13} = e_{45} e_{35}$. If we generate all the $15 \times 32 = 480$ counting functions $C = R_a + R_b + R_c + R_d + R_d + R_e$ that satisfy the previous conditions, that is

$$\begin{cases} e_1 = \pm 1 \\ e_{23} = \pm 1 \\ e_{45} = \pm 1 \\ e_4 = \pm 1 \\ e_{35} = \pm 1 \\ e_{13} = \pm 1 \\ e_1 e_4 e_{13} = e_{45} e_{35} \end{cases}$$

and we limit to the different ones we get 32 design with 11 points and 12 runs, again the same number that has been found in Carlini and Pistone (2007).

6 Conclusions

The problem of decomposing a given fractional factorial design into a disjoint union of regular designs has been discussed. The example of the decomposition of a generic Plackett-Burman design with 12 runs and 5 factors shows that the suggested procedure is actually computable. The same could be done, for example, for latin squares. We expect that the knowledge of such decompositions will improve the understanding of classes of designs of practical interest and possibly assist in the design of block-sequential experiments. It should be pointed out that, from a computational point of view, the generation of all the subgroups of $\mathbb{Z}_2^m$ of a given size could become a critical step as the number m of the factors increases.

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